PROBLEM SET #3, ASTR 451/551

Intensities, Discrete Ordinates, and Lines
due Friday December 2, 2016
23 pts

1. [5 pts] Unlike what you might expect, the full Moon is 8 times as bright as the first or third quarter Moon, not twice as bright. This problem will explore this issue. This is an analytical problem where you should use no outside assistance from computers or integral tables.

Let the intensity of the Sun be $I_\odot$, and suppose the Earth/Moon system is located at a distance $d_\odot$ from the Sun. The Moon has a radius R and is located a distance d from the Earth. In this problem you may assume that both the Sun and Moon subtend small solid angles as seen from the Earth.

Let α be the angle between the Earth and Sun as seen from the Moon, and $F(\alpha)$ be the flux observed at the Earth from the Moon. Define the 'phase function' $j(\alpha) = F(\alpha)/F(0)$. Let the 'albedo' $A = L_{out}/L_{in}$ be the ratio of the energy per second scattered by the Moon at a particular wavelength to the energy per second incident upon the Moon from the Sun at that wavelength.

(a) Show that $A = pq$, where

$$p = \frac{F(0)}{\pi I_\odot \left(\frac{R_\odot}{d_\odot}\right)^2 \left(\frac{R}{d}\right)^2}$$

is called the geometric albedo, and

$$q = 2\int_0^\pi j(\alpha)\sin\alpha d\alpha$$

is called a phase integral.

(b) A 'Lambertian' surface is one that scatters light isotropically from each point. A white piece of paper is a good example – the reflected intensity is essentially independent of the viewing angle. Calculate F(0) for a Lambertian sphere. It might help your thinking to do the problem first for a flat disk whose diameter equals the diameter of the sphere. Strictly speaking, α = 0 corresponds to a lunar eclipse, but we are not interested in modeling this, so ignore the Earth's shadow for this problem.

(c) What are the geometric albedo and phase integral for the Lambertian *sphere* that reflects all incident light? Use the problem to help you give a physical interpretation of the geometric albedo.

(d) Calculate j(π/2) for a sphere that scatters sunlight isotropically. How well does your answer compare with the observed value for the Moon?

2. [10 pts] This problem will help you explore the method of discrete ordinates for the grey atmosphere.

(a) Begin with the integrodifferential radiative transfer equation for the grey atmosphere

$$\mu\frac{dI}{d\tau} - I = -\frac{1}{2}\int_{-1}^{1} I d\mu' \tag{1}$$

and model the integral part with a Gaussian quadrature scheme that has two streams (I_1 and I_2) going up and two complementary streams (I_3 and I_4) going down. Explicitly write out equations for each I_j, j = 1 − 4.

(b) You now need to find four solutions to these four equations. Begin your search for possible solutions by trying solutions of the form

$$I_j = g_j e^{-k\tau} \tag{2}$$

Substitute this solution into your equations in part (a) and derive allowed values for k. If you do it right you should get three numerical solutions. Note that although this is done for you at some level in Collins, I want you to explicitly work through the algebra so you see where everything comes out. Show all steps.

(c) To get the fourth solution, verify that

$$I_j = b(\tau + \mu_j + Q) \tag{3}$$

works, with b and Q arbitrary constants. Write out the final general solution using constants L_i in a form similar to Collins equation 10.2.26, but specifically for your case of 4 streams.

(d) To solve for Q and L_1 Note that the the e^{τ} solution must go to zero because the intensity cannot rise faster than e^{τ} if we are to have a finite flux emerging from the surface. Solve for Q and L_1 by using the fact that at the surface there is no intensity in the downward direction.

(e) With the Eddington approximation we had

$$T^4 = \frac{3}{4} T_e^4 \left[\tau + 2/3\right] \tag{4}$$

With discrete ordinates the 2/3 in the above equation becomes q(τ). Explicitly write out q(τ) for your case of 4 streams. In the limit where $\tau \rightarrow 0$, calculate T/T$_{edd}$, where T$_{edd}$ is the temperature you would get from the Eddington approximation.

3. [8 pts] In this problem you will calculate a spectral absorption line profile given the opacity and making a few simplifying assumptions about the stellar atmosphere. Recall that to do this problem correctly one must follow the discussion of the Milne-Eddington approach, or perhaps just devise a computer code that calculates intensities from the exponential integrals described in the book. However, we can learn a lot about how lines form by doing the following.

We have seen with the Eddington approximation that

$$T^4 = \frac{3}{4} T_e^4 \left(\tau_c + \frac{2}{3}\right) \tag{5}$$

where τ is the continuum optical depth. Hence, at $\tau = 2/3$, the continuum temperature equals the effective temperature. For the purposes of fluxes, we could replace the entire atmosphere with a solid object having the temperature at $\tau = 2/3$.

For a non-grey atmosphere (we want lines after all) let's take the above to be true at all wavelengths; that is, at each wavelength the emergent intensity equals that from a blackbody with a temperature equal to that at $\tau_\lambda = 2/3$. We will assume that there are some lines, but not so many that they affect the overall temperature structure of the atmosphere. In other words, the temperature structure of the atmosphere is set by the optical depth τ_c in the continuum.

The ratio of the opacity in the line to that in the continuum has a triangular profile in wavelength such that the peak line opacity is given by

$$\kappa_l(\lambda_\circ) = \xi_\circ \kappa_c \tag{6}$$

where $\xi_\circ$ is a constant. We will take this constant to be independent of the temperature, so the line/continuum opacity is the same at all T for a fixed λ. So the graph of κ_l/κ_c vs. λ looks like

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     |
xi_0 -               *
     |             *   *
     |           *       *
     |         *           *
   0 -********       |       ************
                   lam_0
     <---------wavelength------------->
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The emergent intensity when κ_l=0 is simply the continuum intensity I_c.

(a) Find a general expression for the intensity ratio I_λ/I_c for a wavelength that has $\kappa_l(\lambda) = \xi\kappa_c$.

(b) Take the star to be similar to the Sun, with an effective temperature 6000 K. Assume the line forms at 1 μm with $\xi_\circ = 1$. Tabulate the spectrum at the points in the figure, that is, where $\kappa_\lambda/\kappa_c = 1.0, 0.75, 0.5, 0.25$, and 0. Sketch the spectrum (I_λ/I_c vs. λ).

(c) Repeat part b, but for a line at 0.4 μm (with the same line/continuum opacity ratio $\xi_\circ$). Compare the depth with the spectrum in part (b), and explain any differences.

(d) What are the deepest possible lines at 1μm and at 0.4μm according to this calculation? Some lines in the Sun are deeper than these, can you explain why?